Gauge invariant Trotterization via shears

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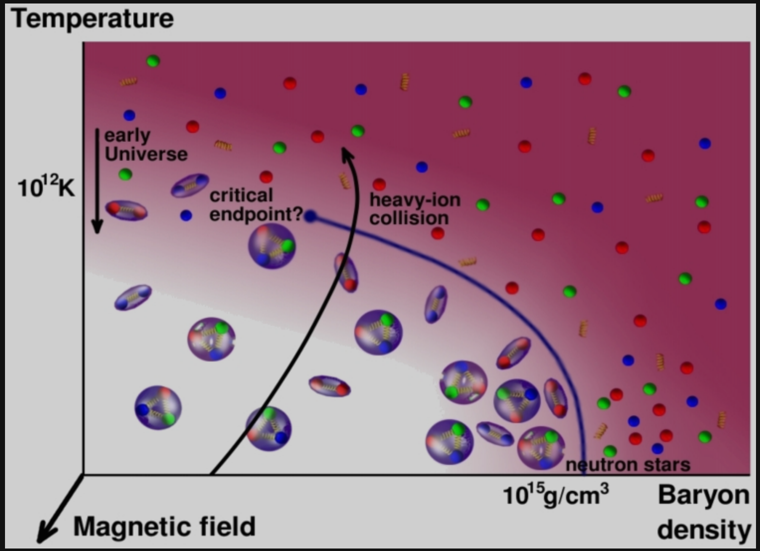
Big picture

Physics targets:

- Simulation of quantum chromodynamics
  - Hadronization
  - Microscopic understanding of nuclear interactions
- Complete phase diagram of QCD
- Equation of state for nuclear matter



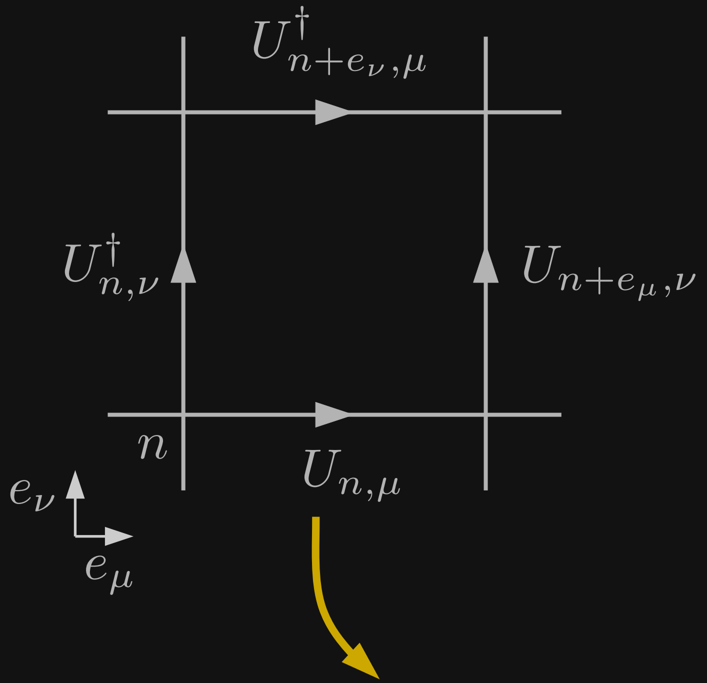
How to make these predictions?

- Nonperturbative problems
  - → Numerically simulate QCD degrees of freedom

Conjectured phase diagram credit: G. Endrődi J.Phys.Conf.Ser. 503 (2014) 012009

Traditional lattice field theory

$x^\mu \to a n^\mu$

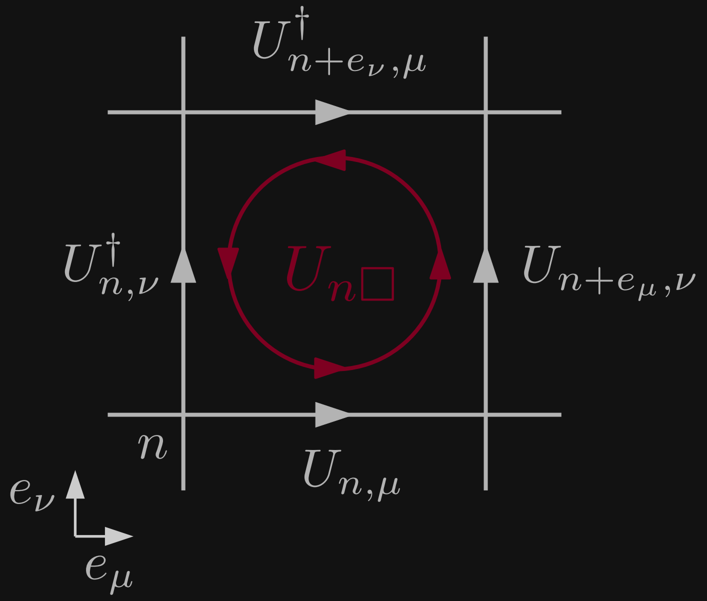


$$\begin{pmatrix} -0.7485 & -0.2744 - 0.6037i \\ 0.2744 - 0.6037i & -0.7485 \end{pmatrix}$$

- Defines a field theory nonperturbatively
- Spacetime discretized with a lattice (e.g. square, cubic, hypercubic)
- Matter particles such as quarks "live" on the sites
- Gauge bosons live on oriented links joining sites
- Gauge fields belonging to some Lie group–the gauge group G

Traditional lattice field theory

$x^\mu \to a n^\mu$



Wilson's gauge action, S_W

"link operators" $U_{n,\mu}$ in gauge group G

$$S_W = -\beta \sum_{n,\mu} \mathrm{tr}\Big(\underbrace{U_{n,\mu} U_{n+e_\mu,\nu} U^\dagger_{n+e_\nu,\mu} U^\dagger_{n,\nu}}_{U_\square} + U^\dagger_\square\Big)$$

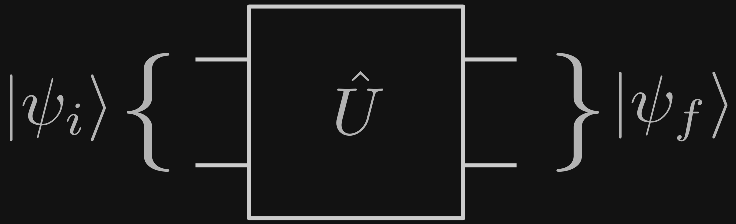
"plaquette" operator

for non-Abelian

- In classical simulations, exp(-S_W) acts like a probability weight for the configuration
- Real-time dynamics and nonzero baryon density both suffer from 'sign problems' in classical simulations

Classical problems.. quantum solutions?

Digital quantum computers:



- Unitary gates: $e^{-it\hat{H}}$ with Hamiltonian of interest
- Want to simulate nonperturbative gauge theory
  - ➔ Gauge theory on the lattice
  - ➔ *Hamiltonian* lattice gauge theory
- Has no apparent sign problems

General problem:

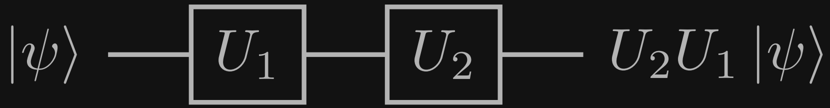
How to map a Hilbert space $\mathcal{H}$, and $\hat{H}$, on to qubits & quantum gates?

Gate-based quantum computing model

Two state, qubit system – computational basis

$$|\uparrow\rangle \leftrightarrow |0\rangle$$
$$|\downarrow\rangle \leftrightarrow |1\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



Two qubit basis: |00>, |01>, |10>, |11>

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Nielsen & Chuang (2001)

Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- An Abelian group, U(1)

$$\langle \phi | q \rangle = \frac{1}{\sqrt{2\pi}} e^{i\phi q}$$

group element or "coordinate" basis for link

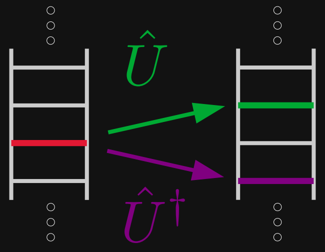
electric representation or "momentum" basis

Quantized with canonical, same-link commutation relations.

$$[E, U] = U$$

$$U \, |q\rangle = |q+1\rangle$$

"U raises E"



Gauge transformations:

$$\hat{U}_{n,i} \rightarrow e^{i(\theta_n - \theta_{n+e_i})} \hat{U}_{n,i}$$

$$\hat{H}_E = \frac{g^2}{2} \sum_{\vec{n},i} \hat{E}^2_{\vec{n},i}$$

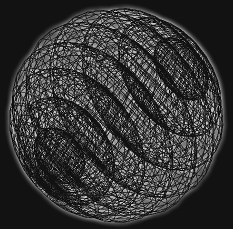
$$\hat{H}_B = - \sum_{\vec{n}} \frac{1}{2g^2} \mathrm{Re}(\hat{U}_{\vec{n},\square})$$

Kogut & Susskind (1975); Creutz (1983), Smit (2002)

Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- Non-Abelian group, e.g. SU(2)



$$\langle g | j, m, n \rangle = \sqrt{\frac{d_j}{|G|}} D^{(j)}_{m,n}(g)$$

group element basis

representation basis

canonical, same-link commutation relations

$$[E^a_{L/R}, E^b_{L/R}] = i f^{abc} E^c_{L/R}$$

$$[E^a_R, U] = U T^a$$

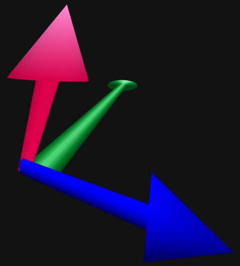
$$[E^a_L, U] = -T^a U$$

Gauge transformations:

$$\hat{U}_{n,i} \to \Omega_n \hat{U}_{n,i} \Omega^\dagger_{n+e_i}$$

"Left" and "right" electric fields to generate the independent left/right rotations.

Left and right electric fields each have 'colored' components in addition to spatial components



True gluons would have 8 such components

Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- Non-Abelian group, e.g. SU(2)

"U adds representations"

$$
\begin{aligned}
U_{m,m'}\left|j,M,M'\right\rangle = \\
&C_+(j,m,m',M,M')\times \\
&\quad\times\left|j+1/2,M+m,M'+m'\right\rangle \\
&+C_-(j,m,m',M,M')\times \\
&\quad\times\left|j-1/2,M+m,M'+m'\right\rangle
\end{aligned}
$$

SU(2) example for the 2x2 link operator

Zohar & Burrello (2015)

Non-Abelian Hamiltonian

$$\hat{H}_E = \frac{g^2}{2}\sum_{n,i}\hat{E}^{\alpha}_{n,i}\hat{E}^{\alpha}_{n,i}$$

$$\hat{H}_B = -\sum_{n}\frac{1}{2g^2}\mathrm{tr}(\hat{U}_{n,\Box}+\hat{U}^{\dagger}_{n,\Box})$$

Hamiltonian lattice gauge theory

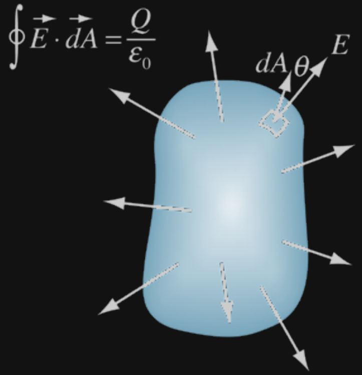
Plus Gauss law constraints

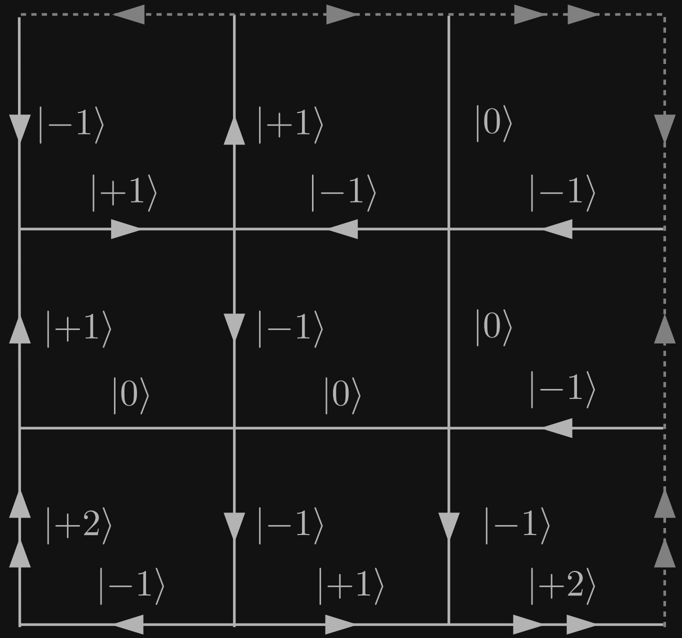
U(1) $\nabla \cdot \mathbf{E} - \rho = 0$

$\hat{\mathcal{G}}_n$ $\quad \rho = \psi^\dagger \psi$

SU(N) $\mathbf{D} \cdot \mathbf{E}^a - \rho^a = 0$

$\hat{\mathcal{G}}^a_n$ $\quad \rho^a = \psi^\dagger T^a \psi$





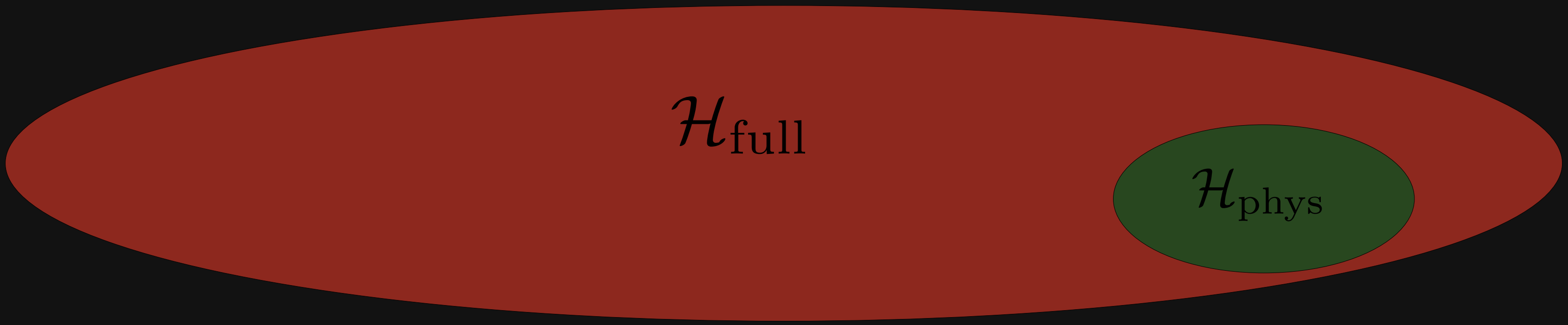
compact U(1)
electric eigenbasis

Gauss's law ⟷ gauge invariance

charge conservation

Potential issues simulating KS formulation

- Qubits wasted on physical states



- Non-Abelian constraints mean individual basis states are virtually never allowed by themselves
- Quantum noise will create components along unphysical directions
- Gauge invariance not necessarily respected by algorithms, even for noiseless simulation

Off-diagonal gauge invariant operators

$$\psi^\dagger(x)\psi(x+1)U(x) \to \psi^\dagger \chi U \to \sigma_\psi^- \sigma_\chi^+ U$$

- Gauge invariant terms, e.g. hopping term, change multiple quantum numbers simultaneously
- Increments on a binary register (*E* register) involve *every* qubit
- Naive Pauli decompositions quickly blow up in number of terms
  - U(1) plaquette w/ two-qubit cutoff: 922 Pauli term
- Approximation via "sub-Trotter steps" liable to have unphysical transitions

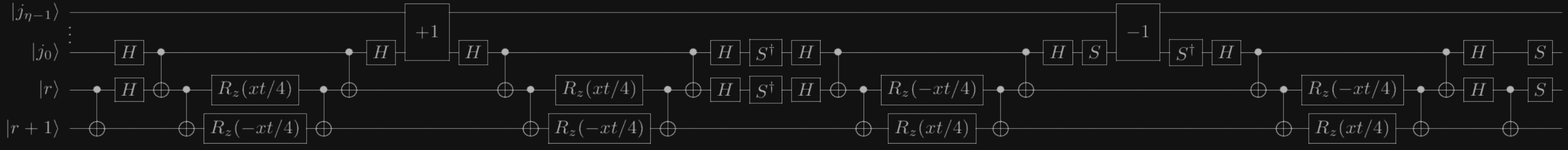
Schwinger model hopping term (2020)

- Schwinger model

Shaw, Lougovski, JRS, Wiebe (2020)

$$|j\rangle = \left|\sum_{n=0}^{\eta-1} j_n 2^n\right\rangle = \bigotimes_{n=0}^{\eta-1} |j_n\rangle$$

$$j = E - E_{\text{min}}$$



Schwinger model hopping term: Shears

$$T_{\mathrm{hop}} = \psi^\dagger \chi U$$

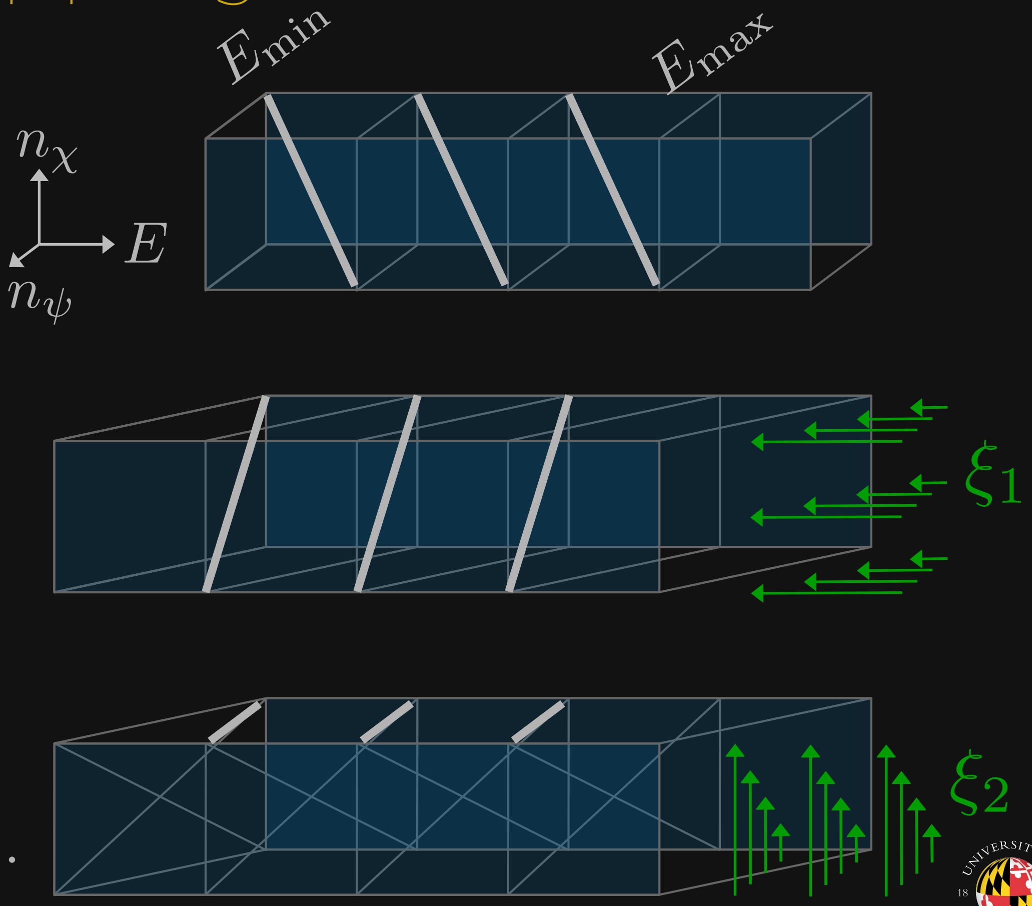
projectors

$$\xi_1 = \delta_{n_\psi,0} + \delta_{n_\psi,1}\lambda^- \ ,$$

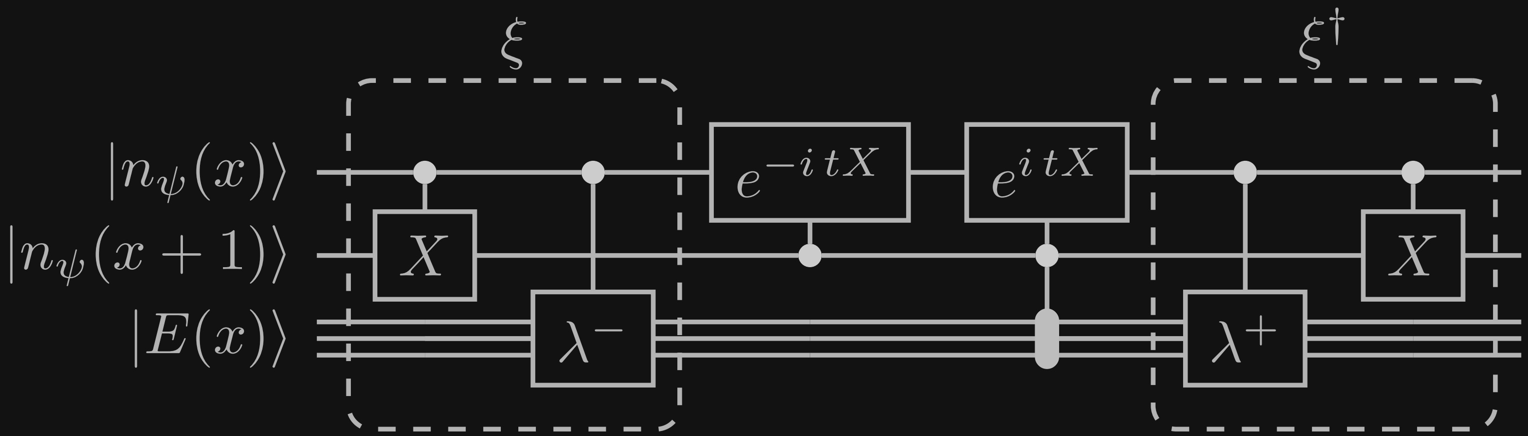
$$\xi_1 T_{\mathrm{hop}} \xi_1^\dagger = \sigma_\psi^- \sigma_\chi^+ (1 - \delta_{E,E_{\mathrm{max}}}) \ .$$

$$\xi_2 = \delta_{n_\psi,0} + \delta_{n_\psi,1} X_\chi \ ,$$

$$\xi_2 (\xi_1 T_{\mathrm{hop}} \xi_1^\dagger) \xi_2^\dagger = \delta_{n_\chi,1} \sigma_\psi^- (1 - \delta_{E,E_{\mathrm{max}}}) \ .$$



Schwinger model hopping term: Circuit



U(1) or Z(N) Plaquette

- Four link plaquette

$$\begin{aligned} U_\Box &= U_0 U_1 U_2^\dagger U_3^\dagger \\ &= \lambda_0^+ \lambda_1^+ \lambda_2^- \lambda_3^- \times \\ &\quad [1-\delta_{N_0,-1}][1-\delta_{N_1,-1}][1-\delta_{N_2,0}][1-\delta_{N_3,0}] \end{aligned}$$

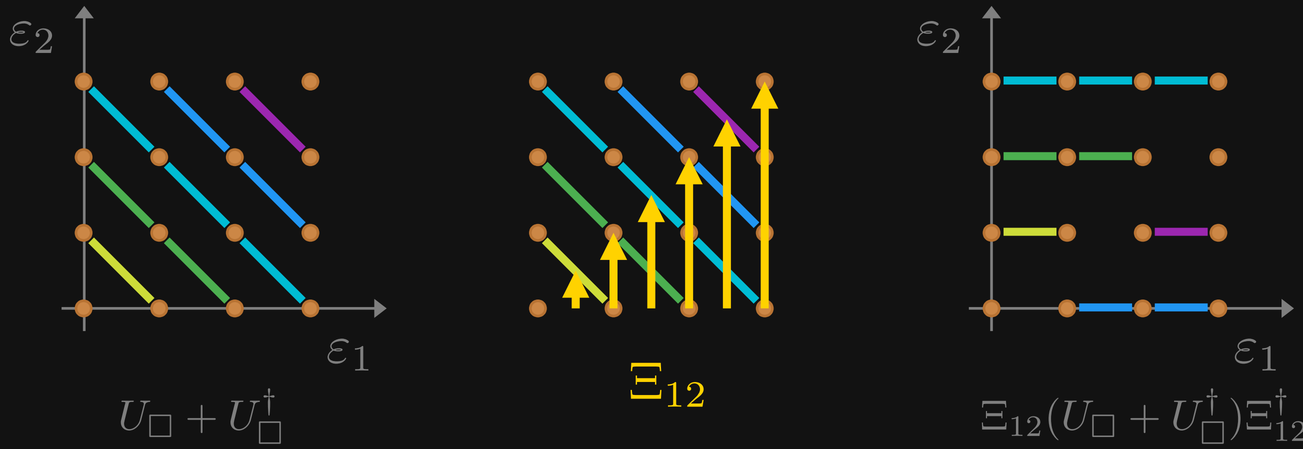
- Off-diagonal on four registers

Toy plaquette

- Consider a two-link "plaquette"

$$U_\square = U_1 U_2^\dagger = \lambda_1^+ \lambda_2^- [1 - \delta_{N_1,-1}][1 - \delta_{N_2,0}] \ .$$

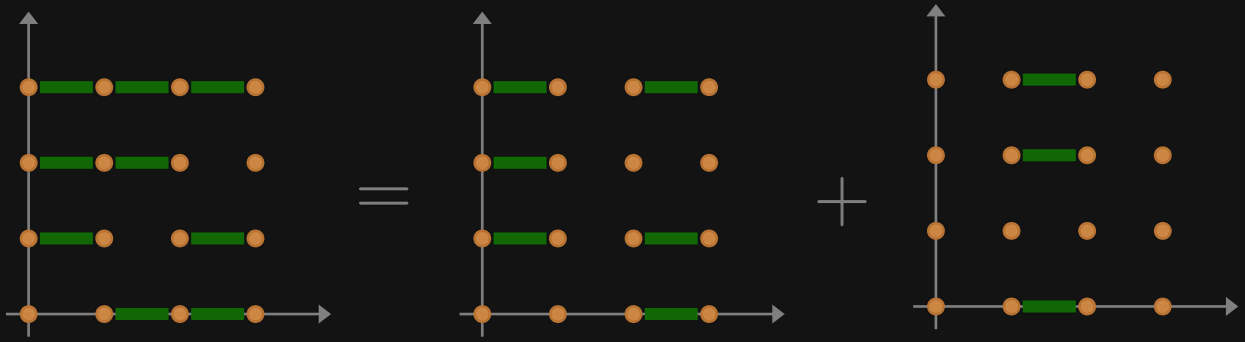
$$\Xi_{12} \equiv \sum_j \delta_{j,\varepsilon_1} (\lambda_2^+)^j \ .$$



$$\Xi_{12} U_\square \Xi_{12}^\dagger = \lambda_1^+ [1 - \delta_{N_1,-1}][1 - \delta_{N_2,N_1}].$$

Toy plaquette

- The couplings can be expressed as a sum of two terms



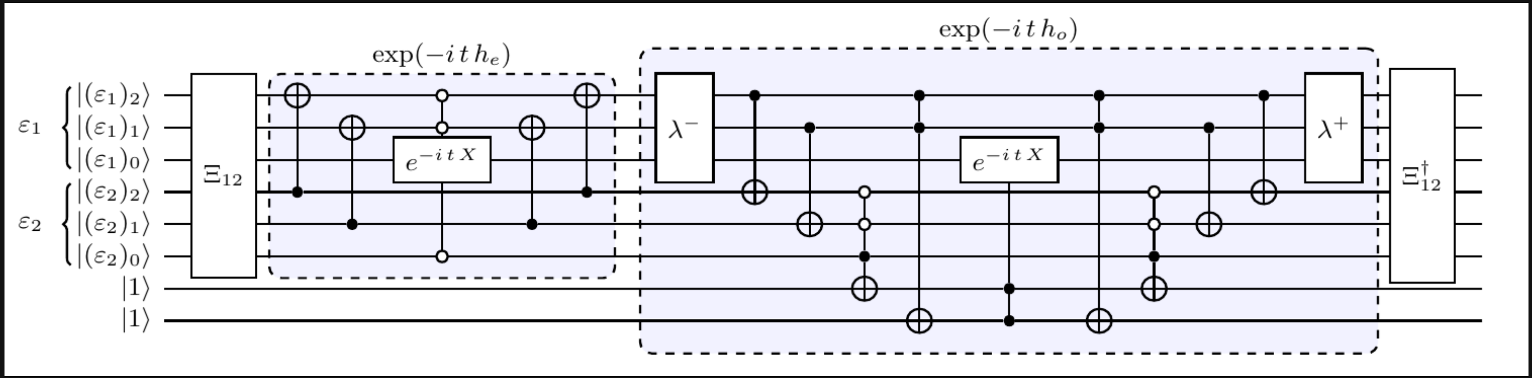
$\Xi_{12}(U_\square + U_\square^\dagger)\Xi_{12}^\dagger \qquad X_{1,\mathrm{lsb}}\, P_e \qquad \lambda_1^+[X_{1,\mathrm{lsb}}\, P_o]\lambda_1^-$

$$U_\square + U_\square^\dagger \xrightarrow{\Xi_{12}} h_e + h_o \ ,$$

$$h_e \equiv \sigma_{1,\mathrm{lsb}}^-[1-\delta_{N_1,-1}][1-\delta_{N_2,N_1}] + [1-\delta_{N_1,-1}][1-\delta_{N_2,N_1}]\sigma_{1,\mathrm{lsb}}^+ \ ,$$

$$h_o \equiv \lambda_1^+\sigma_{1,\mathrm{lsb}}^-\lambda_1^-[1-\delta_{N_1,-1}][1-\delta_{N_2,N_1}] + [1-\delta_{N_1,-1}][1-\delta_{N_2,N_1}]\lambda_1^+\sigma_{1,\mathrm{lsb}}^+\lambda_1^- \ .$$

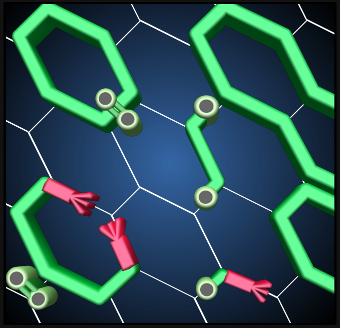
Toy plaquette: Circuit



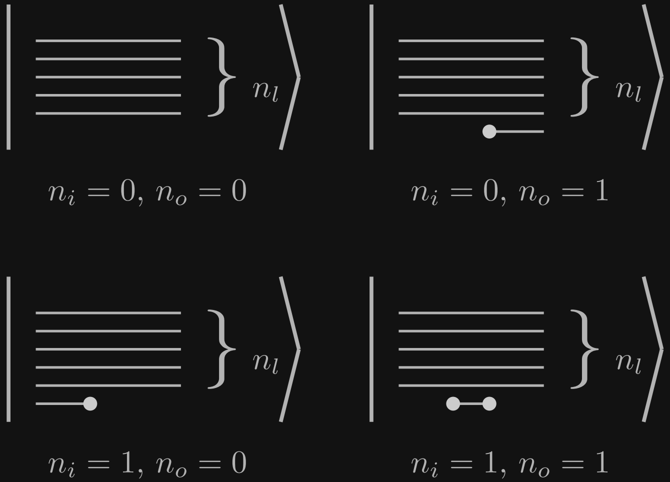
Outlook

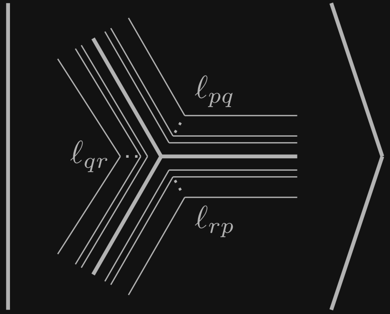
- The true, four-link plaquette is solved using shears in a 4D space
- Shearing appears to work when gauge constraints are *simultaneously diagonalizable*
  - Can characterize each matrix element of H as an allowed or unallowed transition
  - Off-diagonal, product of ladder operators + shearing → Controlled ladder operator on one quantum number
  - Can use sub-Trotter steps safely as long as we reproduce all the allowed transitions
- Non-Abelian? SU(2)?
  - Kogut-Susskind – unlikely to help
  - *Loop-String-Hadron* has Abelian (commuting) constraints

Loop-String-Hadron



- Loop-String-Hadron has formal lattice consisting of 'quark sites' and 'gluonic sites'
  - Quark sites have a local basis $|n_i, n_o, n_l\rangle$
  - Gluonic sites have a local basis $|\ell_{qr}, \ell_{rp}, \ell_{pq}\rangle$





$\sim |\ell_{pq}, \ell_{qr}, \ell_{rp}\rangle$

Constraints known as "Abelian Gauss Law": Abelian flux must be conserved along each link

Raychowdhury & Stryker (2020)

Summary

- Off-diagonal terms such as hopping terms in Schwinger model or plaquette operators in d>1 U(1) gauge theories require correlated changes of quantum numbers
- Shears can help change basis such that off-diagonal operators change one quantum number only
  - Other registers may still be involved — as controls
- Commuting constraints and Cartesian 'space of quantum numbers' seem key
- Non-Abelian will no doubt be harder, but there is reason to be optimistic

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