Quantum simulating non-Abelian lattice gauge theories:
gauge invariance, point splitting, and magnetic interactions

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ECT* workshop
“Nuclear and particle physics on a quantum computer: where do we stand now?”

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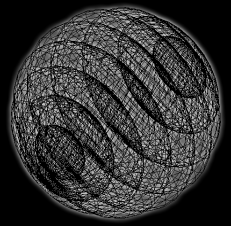
Hamiltonian lattice gauge theory

- Temporal gauge, continuous-time limit → Kogut-Susskind Hamiltonian formulation
- Gauge fields on spatial links with on-link Hilbert spaces
- E.g., SU(2)

Phys. Rev. D 11, 395 (1975)

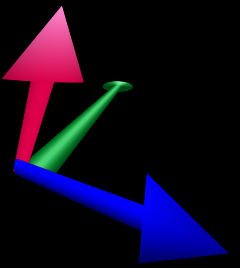


$$\langle g | j, m, m' \rangle = \sqrt{\frac{d_j}{|G|}} D^{(j)}_{m,m'}(g)$$

group-element basis

irrep basis

Left and right electric fields each have color-charge components, in addition to spatial components



Gauge transformations: $\hat{U}_{n,i} \to \Omega_n \hat{U}_{n,i} \Omega^\dagger_{n+e_i}$

- Rotations from the left (Ω_n) and right (Ω_{n+ei}) are generated by “left” and “right” electric fields

$$[\hat{E}^\alpha_{L/R}, \hat{E}^\beta_{L/R}] = i f^{\alpha\beta\gamma} \hat{E}^\gamma_{L/R}$$
$$[\hat{E}^\alpha_R, \hat{U}_{mm'}] = \left(\hat{U} T^\alpha\right)_{mm'}$$
$$[\hat{E}^\alpha_L, \hat{U}_{mm'}] = -\left(T^\alpha \hat{U}\right)_{mm'}$$

canonical commutation relations for a link

Hamiltonian lattice gauge theory

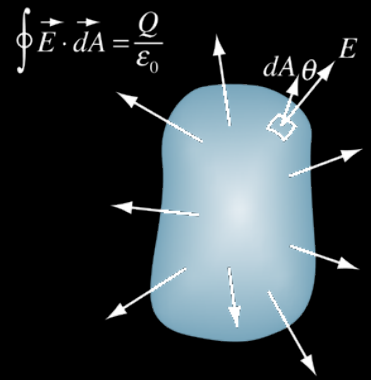
Plus Gauss law constraints

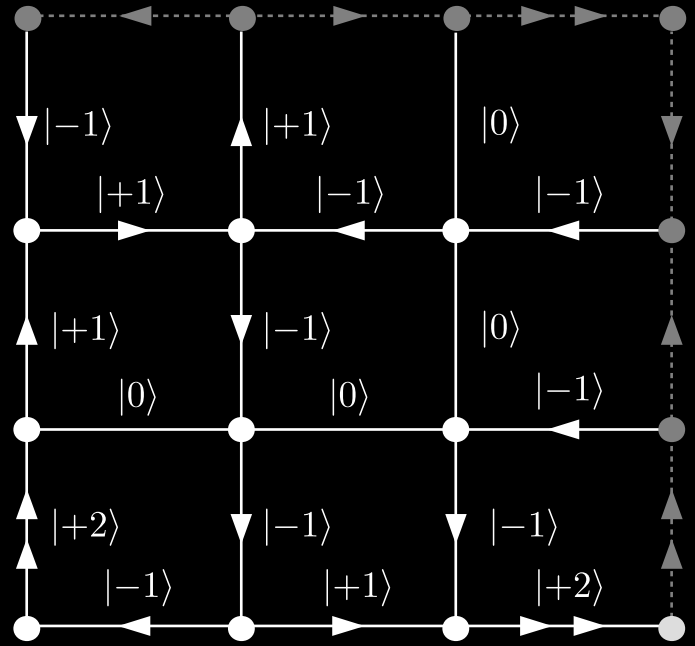
U(1) $\nabla \cdot \mathbf{E} - \rho = 0$

$\hat{\mathcal{G}}_n$ $\rho = \psi^\dagger \psi$

SU(N) $\mathbf{D} \cdot \mathbf{E}^a - \rho^a = 0$

$\hat{\mathcal{G}}_n^a$ $\rho^a = \psi^\dagger T^a \psi$





compact U(1)
electric eigenbasis

Gauss's law ⟷ gauge invariance

charge conservation

Hamiltonian lattice gauge theory

A feel for the on-link operators and states of SU(2):

$$E_R^3 \left|j, M, M'\right\rangle = M' \left|j, M, M'\right\rangle$$
$$E_R^\alpha E_R^\alpha \left|j, M, M'\right\rangle = j(j+1) \left|j, M, M'\right\rangle$$

$$U_{m,m'} \left|j, M, M'\right\rangle = C_+(j, m, m', M, M') \left|j + 1/2, M + m, M' + m'\right\rangle + C_-(j, m, m', M, M') \left|j - 1/2, M + m, M' + m'\right\rangle$$

Off-diagonal SU(2) link operator

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^\alpha_{n,i} \hat{E}^\alpha_{n,i}$$

$$\hat{H}_B = -\sum_n \frac{1}{2g^2} \mathrm{tr}(\hat{U}_{n,\square} + \hat{U}^\dagger_{n,\square})$$

$$H_I = \sum_{n,i} \hat{\psi}^\dagger(n) \hat{U}_{n,i} \hat{\psi}(n + e_i) + \mathrm{H.c.}$$

Formulation, basis considerations

- Different bases/formulations → different costs
- For gauge theories in particular, numerous *formulations*

- Kogut-Susskind formulation
  - Irrep/"angular momentum" basis
    Byrnes, Yamamoto, Zohar, Burrello, et al.
  - Group-element basis *Zohar, NuQS collab., et al.*
- Gauge magnets/quantum link models
  Wiese, Chandrasekharan, et al.
- Tensor lattice field theory
  Meurice, Sakai, Unmuth-Yockey, et al.
- Dual/rotor formulations *Kaplan, **JRS**, Haase, Dellantonio, et al., Bauer, Grabowska, Kane*
- Casimir variables / "local-multiplet basis" *Klco, Savage, **JRS**, Ciavarella*
- Purely fermionic formulations (1+1D & OBC)
  Muschik, Atas, Zhang, IQuS@UW group, Powell, et al.
- Prepotential/Schwinger boson formulations
  Mathur, Anishetty, Raychowdhury, et al.

- Loop-string-hadron formulation
  *Raychowdhury, **JRS**, Davoudi, Shaw, Dasgupta, Kadam*
- Light-front formulation
  Kreshchuk, Kirby, Love, Yao, et al.
- Qubit models *Chandrasekharan, Singh, et al.*
- *q*-deformed Kogut-Susskind
  Zache, González-Cuadra, Zoller
- Scalar field theory…
  - Harmonic oscillator basis
    Klco & Savage
  - Single-particle basis
    Barata, Mueller, Tarasov, Venugopalan
  - Future gauge-field generalizations??

… Not a complete list!

Choice of basis

Most common basis choice: **Electric/irrep**

Electric-basis pros

- States naturally discretized (for compact Lie groups)
- Gauss's law a function of electric fields
- Natural "UV" truncation scheme
  - Easily translates to truncating operators

Electric-basis cons

- Better-suited to strong coupling (opposite of continuum QCD)
- Many off-diagonal operators in 3+1 Hamiltonian

Choice of basis

Group-element basis pros

- Link operators are diagonalized
- No Clebsch-Gordon coefficients
- Well-suited for weak-coupling limit

A detail of Spinoza monument in Amsterdam. © Dmitry Feichtner-Kozlov

Group-element basis cons

- Limited number of regular subgroups for SU(N)
  - Limited “resolution” with subgroups
  - 120 elements for SU(2)
  - 1080 for SU(3) [NuQS collab.]
- Sub*sets* generally do not preserve gauge symmetry
- Electric fields become tricky

Digital quantum simulation: Time evolution

Trotterization: Evolve for time *t* in *s* steps,

$$e^{-i\,t\,H} = \left(e^{-i\,\frac{t}{s}\,H}\right)^{s}$$

Product formulas: Approximate exponential of a sum by product of exponentials

$$e^{-i\,\delta t\,\sum_k H_k} \simeq \prod_k e^{-i\,\delta t\,H_k}$$

Simplest case: Same ordering of H_k in every step
Generalizations: Higher-order Trotter; randomized

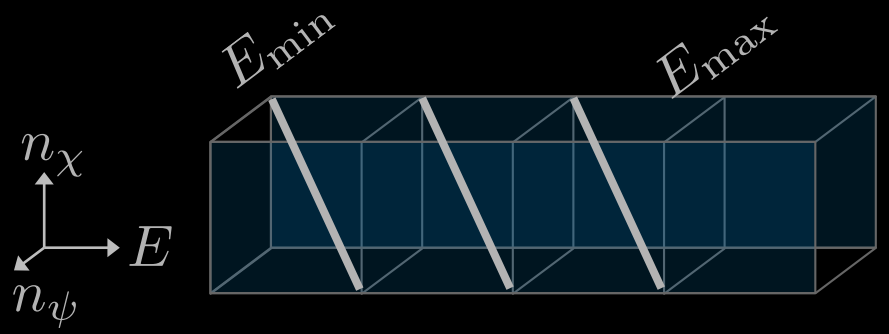
The art: *Finding good H_k that we know how to circuitize*

About the H_k

- Diagonal H_k are straightforward (possibly expensive)
- Could always take H_k to be the Pauli operator basis (naive Pauli decomposition) but this is a massive number of rotations
- More H_k → more subroutines & more Trotter error
- H_k can break Gauss's law
  - How to systematically conserve Gauss's law?

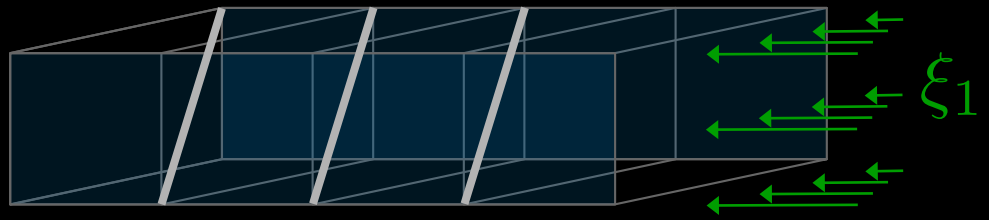
Schwinger model hopping terms: Sheared

$$T_{\text{hop}} = \psi^\dagger \chi U$$

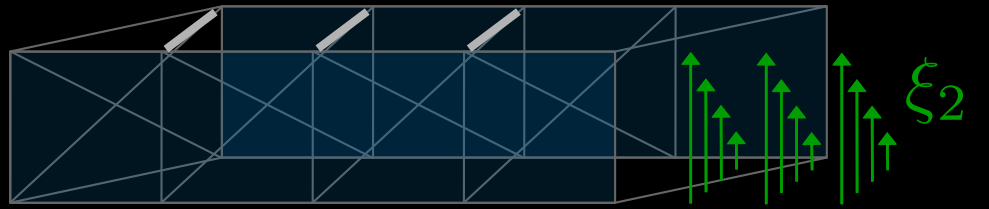


projectors

$$\xi_1 = \delta_{n_\psi,0} + \delta_{n_\psi,1}\lambda^- \ ,$$
$$\xi_1 T_{\text{hop}} \xi_1^\dagger = \sigma_\psi^- \sigma_\chi^+ (1 - \delta_{E,E_{\text{max}}}) \ .$$



$$\xi_2 = \delta_{n_\psi,0} + \delta_{n_\psi,1} X_\chi \ ,$$
$$\xi_2 (\xi_1 T_{\text{hop}} \xi_1^\dagger) \xi_2^\dagger = \delta_{n_\chi,1} \sigma_\psi^- (1 - \delta_{E,E_{\text{max}}}) \ .$$

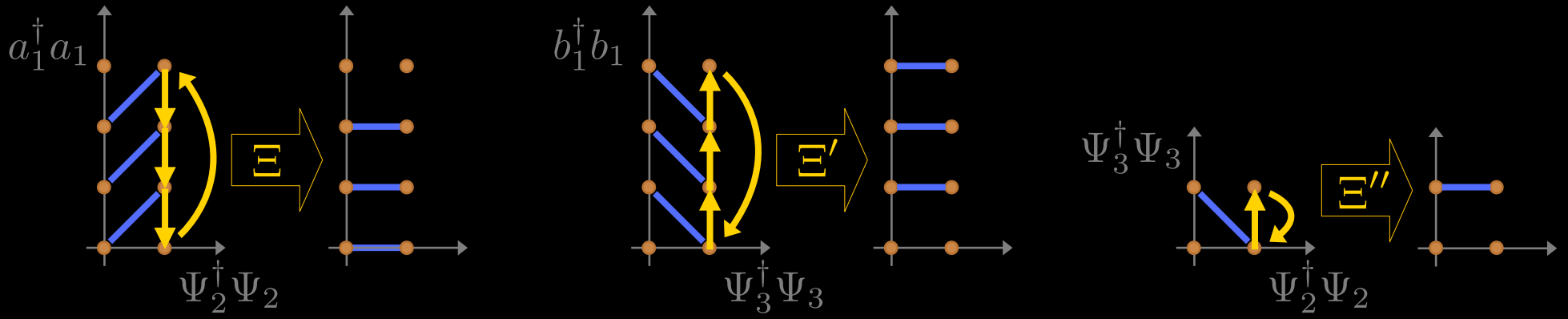


JRS. [2105.11548]

SU(2) hopping terms (Schwinger boson form.)

Three shears to prepare for rotation:

$$H^{(8)}_{\text{hop}}/x = -\sigma_2^+ A_1 \sigma_3^- b_1 + \text{H.c.}$$



$$-\sigma_2^+ A_1 \sigma_3^- b_1 + \text{H.c.} \xrightarrow{H_2 \Xi'' \Xi' \Xi} -Z_2 \, |1\rangle\langle 1|_3 \sqrt{\frac{(a_1^\dagger a_1) b_1^\dagger b_1}{(a^\dagger \cdot a)(a^\dagger \cdot a + 1)}}$$

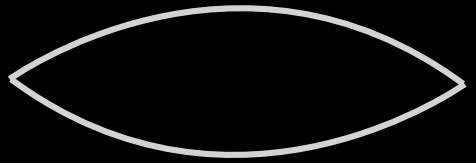
- *Z*-rotation on fermionic qubit 2, controlled by fermionic qubit 3, with phase depending on three occupation numbers of a_1, a_2, b_1

Z. Davoudi, A.F. Shaw, & JRS (2022)
under review at Quantum

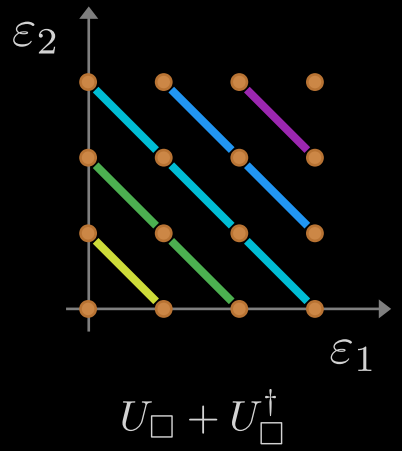
Toy U(1) or Z(N) plaquette

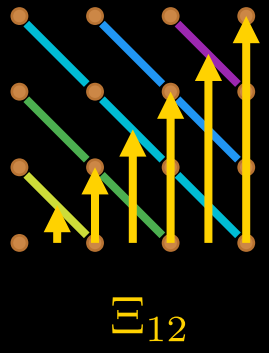
- Consider a two-link “plaquette”

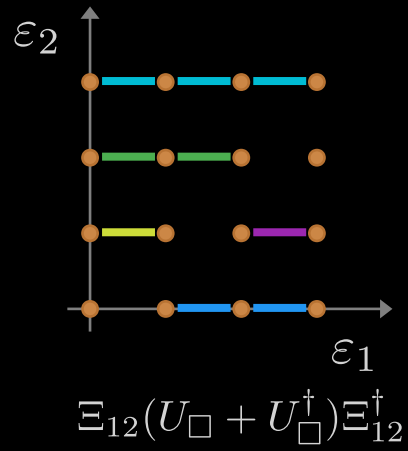
$$U_\square = U_1 U_2^\dagger = \lambda_1^+ \lambda_2^- [1 - \delta_{N_1,-1}][1 - \delta_{N_2,0}] \ .$$



$$\Xi_{12} \equiv \sum_j \delta_{j,\varepsilon_1} (\lambda_2^+)^j \ .$$



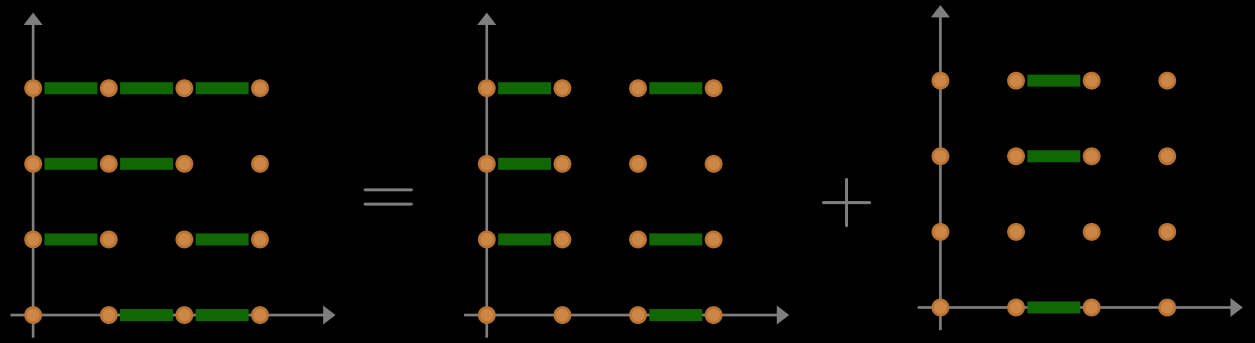




$$\Xi_{12} U_\square \Xi_{12}^\dagger = \lambda_1^+ [1 - \delta_{N_1,-1}][1 - \delta_{N_2,N_1}].$$

Toy plaquette

- The couplings can be expressed as a sum of two terms



$$\hat{U}_1\hat{U}_2^\dagger + \hat{U}_1^\dagger\hat{U}_2 \xrightarrow{\Xi_{12}} \hat{h}_e + \hat{h}_o\,,$$

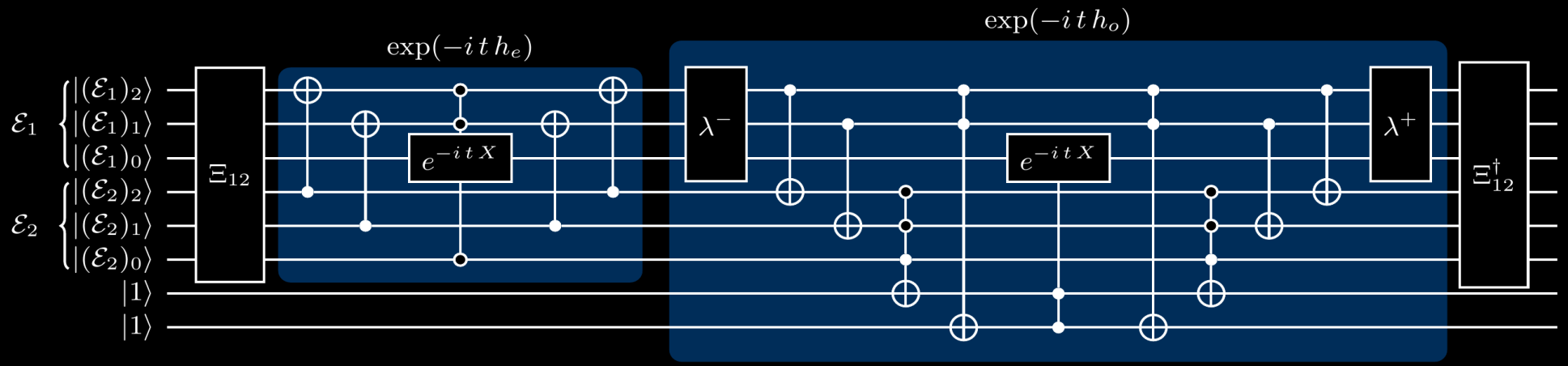
$$\hat{h}_e = \hat{\pi}_e X_1^{(\mathrm{lsb})},$$

$$\hat{\pi}_e \equiv (1 - \delta_{\hat{\mathcal{E}}_2, 2\lfloor \hat{\mathcal{E}}_1/2 \rfloor}),$$

$$\hat{h}_o = \lambda_1^+ \hat{\pi}_o X_1^{(\mathrm{lsb})} \lambda_1^-,$$

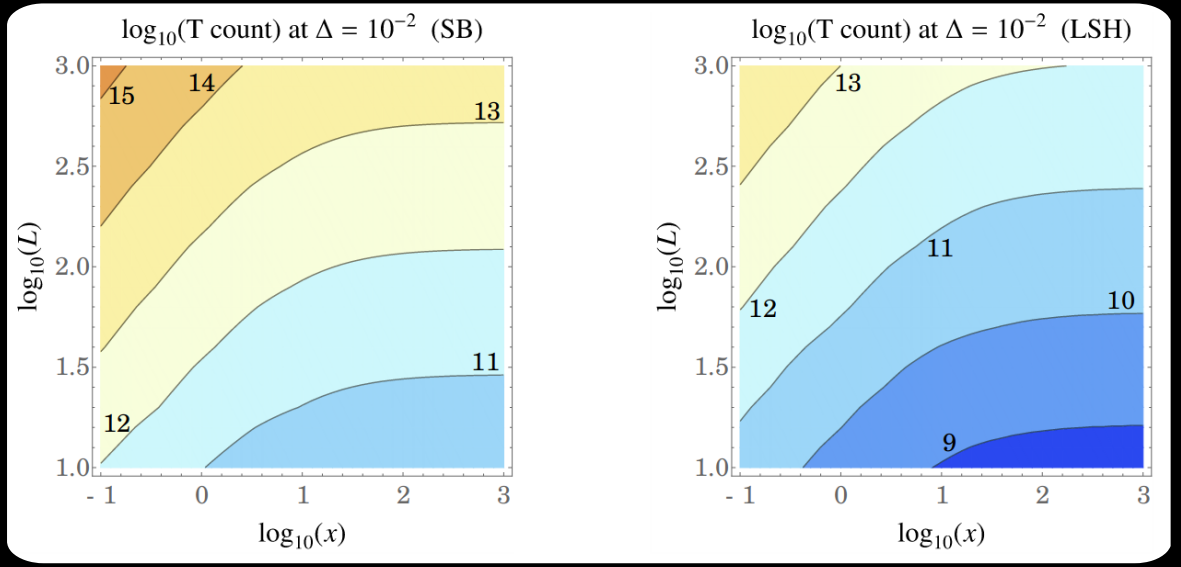
$$\hat{\pi}_o \equiv (1 - \delta_{2\lfloor \hat{\mathcal{E}}_1/2 \rfloor + 1, -1})(1 - \delta_{\hat{\mathcal{E}}_2, 2\lfloor \hat{\mathcal{E}}_1/2 \rfloor + 1}).$$

Toy plaquette: Circuit



True plaquette

- Full plaquette induces moves by ±(1,1,-1,-1) in the 4D space of electric quantum numbers
- Shear in 01-plane
  → ±(0,1,-1,-1)
- Shear in 23-plane
  → ±(0,1,-1,0)
- Shear in 12-plane
  → ±(0,1,0,0)
- Internal arithmetic needed to avoid cutoff-wrapping effects (more projectors)



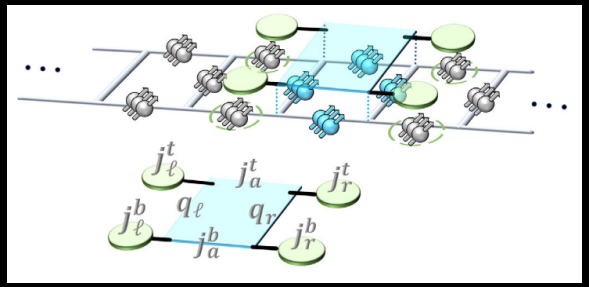
Note: The values of nonzero matrix elements are irrelevant. Procedure applies also to non-Abelian! (WIP)

Combining irreps at a vertex

Klco, Savage, JRS (2020)
Ciavarella, Klco, Savage (2021)

- Another approach to plaquettes: “Controlled-plaquette operators”
  - First sum over states around a vertex to remove the J_z component/isospin/hypercharge quantum numbers
  - Easiest to work out for 3-point vertices
  - Leftover constraints: Triangle inequalities/generalizations
  - At 4- and 6-point vertices: irreps around a vertex don’t fully specify state

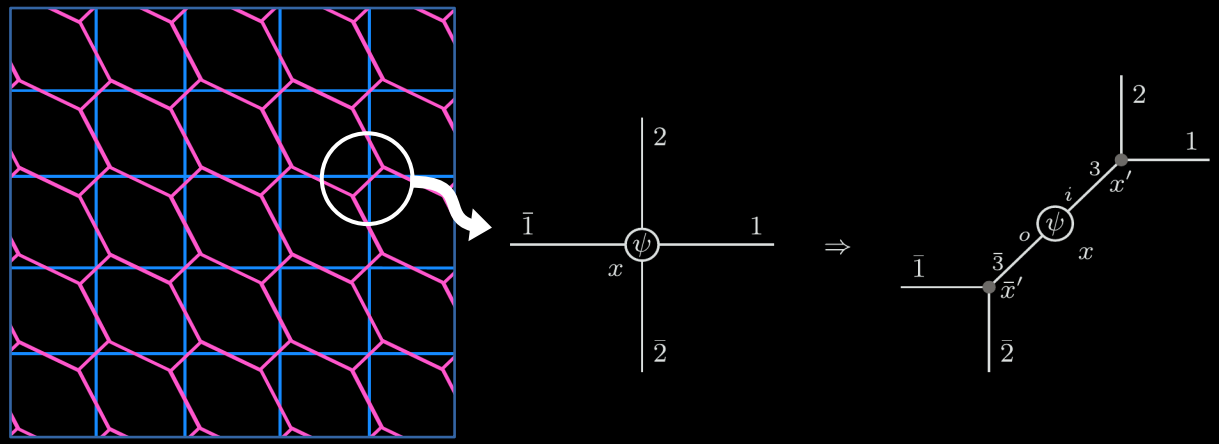
    (Ex: ½ x ½ x ½ x ½ = 0+0+1+1+1+2)
    - There has to be right number of DOF per site, link irreps aren’t enough



- Classically compute matrix elements between different irrep configurations
  - Can be a lot of two-state rotations

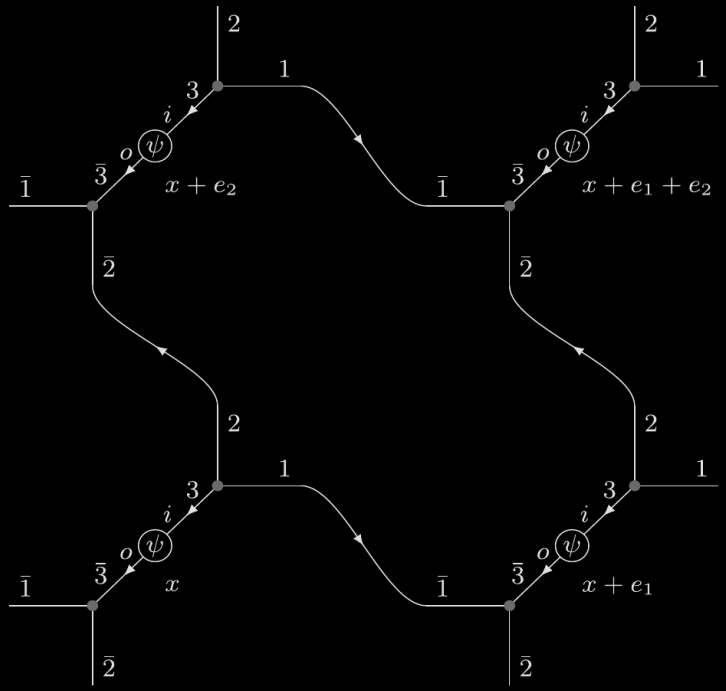
Point splitting

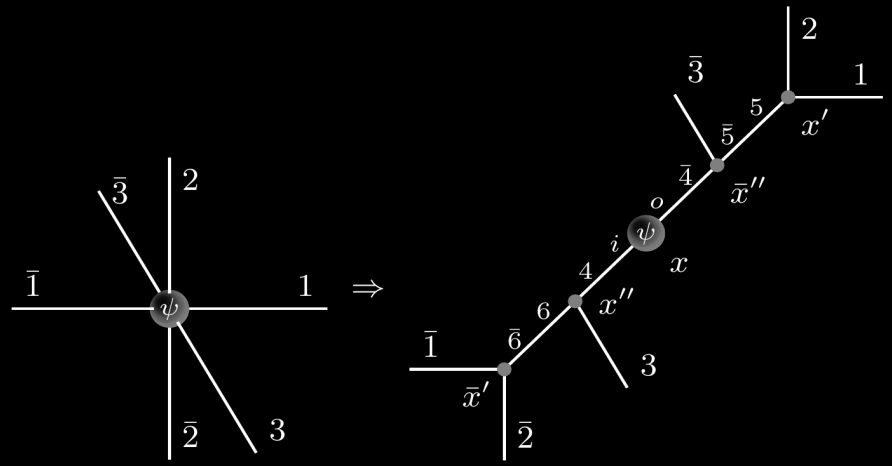
Point-splitting (I. Raychowdhury, 2019) distributes the vertex DOF to “virtual links” so that one set of quantum numbers specifies a unique state
There are multiple equivalent choices to point split a site. 2D and 3D SU(2) already worked out (Raychowdhury & JRS, 2019) and SU(3) in the works.



I. Raychowdhury, Eur. Phys. J. C ‘19

Point splitting





3D site

Summary

- There **are** options for gauge-invariant DOF
  - Vertices inherit more structure when Hilbert space is reduced
- We have ways to deal with plaquette operators AND maintain gauge invariance via shears
  - Major cost: non-Abelian coefficients. Groups are still learning how to best deal with vertices.
  - Point-splitting gives a geometrical representation and concrete book-keeping device to what is really just the necessary CSCO. Stay tuned for SU(3)!

FIN

Thank you for your attention!